Lezione 10 c1

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Istituzioni di Matematiche
CdL Scienze Biologiche

Esercizio

Studiare inf e sup del seguente insieme

$$A = \left\{ 5^{\frac{n+1}{n}} : n \geq 1 \right\}$$

Notiamo che $a^x > 0 \quad \forall x \in \mathbb{R}$ (con $a>0$)

In particolare gli elementi di A sono numeri positivi

Cerco maggioranti per A, $\boxed{x>0}$

$$5^{\frac{n+1}{n}} \leq x \qquad \forall n \in \mathbb{N}^+$$

$$\overset{5>1}{\iff} \quad \log_5 5^{\frac{n+1}{n}} \leq \log_5 x$$

$$\iff \quad \frac{n+1}{n} \leq \log_5 x$$

$$\iff \quad 1 + \frac{1}{n} \leq \log_5 x$$

$$\Longleftrightarrow \quad \boxed{\frac{1}{n} \le \log_5 x - 1} \quad ⊛$$

se $\log_5 x - 1 < 0 \Longrightarrow$ mai verificata

ossia non può essere maggiorante

$$\Updownarrow$$

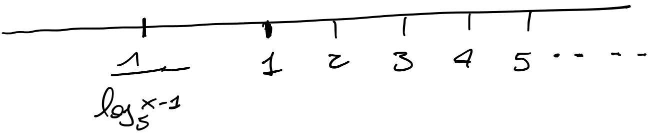
$$\log_5 x < 1$$

$$\Updownarrow$$

$$\boxed{x < 5} \longleftrightarrow \text{NO MAGG.}$$

Sia $\boxed{x \ge 5}$ Da ⊛ posso passare ai reciproci, per ottenere

$$\boxed{n \ge \frac{1}{\log_5 x - 1}} \quad \forall n \in \mathbb{N}^+$$



$$\boxed{x \ge 5} \Longrightarrow \quad \frac{1}{\log_5 x - 1} \le 1$$

$$\iff \quad \log_5 x - 1 \geq 1$$

$$\log_5 x \geq 2$$

$$\iff \quad 5^{\log_5 x} \geq 5^2$$

$$\iff \quad \boxed{x \geq 25}$$

$$\implies \quad \sup A = \min A^* = 25$$

Problema. $25 \in A$?

Ossia $25 = 5^{\frac{n+1}{n}}$ pr qualche $n \geq 1$?

$$5^2 = 5^{\frac{n+1}{n}}$$

$$2 = \frac{n+1}{n}$$

$$2n = n+1$$

$$2n - n = 1$$

$$n = 1 \qquad \text{Sì}$$

$$\Rightarrow \quad \boxed{25 = \max A}$$

Studiamo l'insieme dei minoranti

Ricordo che gli elementi di A sono positivi ($0 \in A_*$)

Cerco minorante $y > 0$ (se c'è!)

$$y \le 5^{\frac{n+1}{n}} \qquad \forall n \in \mathbb{N}^+$$

$$\log_5 y \le \log_5 5^{\frac{n+2}{n}}$$

$$\log_5 y \le \frac{n+1}{n}$$

$$\log_5 y \le 1 + \frac{1}{n}$$

$$\boxed{\log_5 y - 1 \le \frac{1}{n}} \qquad (**)$$

se $\boxed{\log_5 y - 1 < 0} \Longrightarrow$ (**) è sempre verificata (OK)

$\Updownarrow$

($y < 5$)

Verifico se ci sono minoranti ($y \geq 5$)

Da (**) passo ai reciproci

$$\boxed{\frac{1}{\log_5 y - 1} \geq n} \qquad \forall n \in \mathbb{N}^+$$

Ossia ($y \geq 5$) è minorante se e solo se ogni numero naturale ≥ 1 è più piccolo del numero $\frac{1}{\log_5 y - 1}$

$\Longrightarrow \mathbb{N}^+$ limitato superiorm. $\neq$

$\Longrightarrow y \geq 5$ non può essere minorante

Ossia

$$A_* = \{ y \in \mathbb{R} : y \leq 5 \}$$

$$\Rightarrow \quad \inf A = \max A_* = 5$$

Problema $\quad 5 \in A$?

Ossia $\quad 5 = 5^{\frac{n+1}{n}}$ per qualche n ?

$$1 = \frac{n+1}{n} \quad \text{per qualche } n \in \mathbb{N} \quad n \geq 1$$

$$n = n+1$$

$$n - n = 1$$

$$0 = 1 \quad \text{↯}$$

Concludo che $5 = \inf A$ ma 5 non è $\min A$ ∎

Esercizio Determinare inf e sup

dell'insieme

$$A = \left\{ (-1)^n \cdot \frac{n-1}{n} : n \in \mathbb{N}^+ \right\}$$

Scrivo $A = A_p \cup A_d$

dove $A_p = \left\{ \frac{n-1}{n} : n \in \mathbb{N}^+ \text{ pari} \right\}$

$A_d = \left\{ -\frac{n-1}{n} : n \in \mathbb{N}^+ \text{ dispari} \right\}$

Cerco inf e sup di A_p

Noto che $\frac{n-1}{n} > 0$ $\forall n \geq 2$ pari

Cerco maggioranti $\boxed{x > 0}$

$$\frac{n-1}{n} \leq x \quad \forall n \in \mathbb{N}^+ \text{ pari}$$

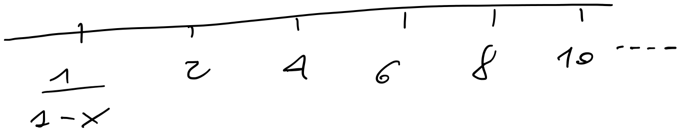
$$n - 1 \leq x \cdot n$$

$$\boxed{(1-x) \cdot n \leq 1} \quad (*)$$

se $\boxed{1-x<0}$ $(x>1)$ divido per $1-x$

$$\boxed{n \geq \frac{1}{1-x}} \quad \forall n \in \mathbb{N}^+$$

n pari



Caso $x>1$

$$\iff \frac{1}{1-x} \leq 2$$

è negativo

sempre verificato

Ossia ogni $x>1$ è maggiorante per A

Cerco maggioranti $\boxed{x \leq 1}$ $(1-x \geq 0)$

Riparto da

$$(1-x)\cdot n \leq 1$$

$$n \le \frac{1}{1-x} \quad \forall n \in \mathbb{N}^+ \text{ pari}$$

$\mathbb{N}^+$ pari sono limitati superiormente.

$\not\equiv$

Ossia gli elementi $x<1$ non possono essere maggioranti per A_p

$$\Rightarrow \sup A_p = \min A_p^* = 1$$

Problema $1 \in A_p$?

ossia $1 = \frac{n-1}{n}$ per qualche n pari?

$$\cancel{n} = \cancel{n} - 1$$

$$0 = -1$$

$\boxed{\text{mai}}$

Quindi 1 non può essere max A_p

Studio minoranti di A_P

Già sappiamo che $0 \in (A_P)_*$

Cerco minorante $y > 0$

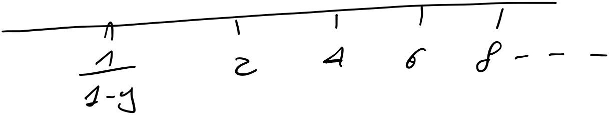
$$y \le \frac{n-1}{n} \quad \forall n \text{ pari}$$

$$y \cdot n \le n-1$$

$$\boxed{1 \le (1-y) \cdot n} \quad (**)$$

Se $y < 1$ ossia $1-y > 0$

Da $(**)$ $$\frac{1}{1-y} \le n \quad \forall n \in \mathbb{N}^+ \text{ pari}$$



$$\Longleftrightarrow \quad \frac{1}{1-y} \leq 2 \qquad \boxed{1-y>0}$$

$$1-y \geq \frac{1}{2}$$

$$y-1 \leq -\frac{1}{2}$$

$$y \leq 1-\frac{1}{2}$$

$$\boxed{y \leq \frac{1}{2}} \qquad \boxed{\text{OK}}$$

Se invece $\boxed{y>1}$ Da $\boxed{**}$

$$\frac{1}{1-y} \geq n \qquad \forall n \in \mathbb{N}^+ \text{ pari}$$

Mai perché $\mathbb{N}^+$ pari non è lim sup

Concludo $(A_p)_* = \{ y \in \mathbb{R} : y \leq \frac{1}{2} \}$

$$\Longrightarrow \quad \inf A_p = \max (A_p)_* = \frac{1}{2}$$

Problema $\frac{1}{2} \in A_p$?

Ossia $$\frac{1}{2} = \frac{n-1}{n}$$

$$n = 2n - 2$$

$$2 = 2n - n$$

$$\boxed{2 = n}$$ Si

$$\Rightarrow \boxed{\frac{1}{2} = \min A_p}$$

Studio $$A_d = \left\{ -\frac{n-1}{n} : n \in \mathbb{N}^+ \text{ disp} \right\}$$

Sup Essendo gli elementi di A_d sono tutti negativi

Ossia $0 \in A_d^*$

cerco maggioranti $\boxed{x < 0}$

$$-\frac{n-1}{n} \le x \qquad \forall n \in \mathbb{N}^+ \text{ disp}$$

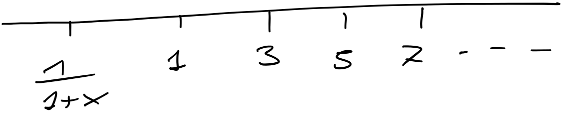
$$-(n-1) \le x \cdot n$$

$$-n+1 \le x \cdot n$$

$$\boxed{1 \le (1+x)\,n} \quad (*)$$

se $1+x>0$ $\quad (x>-1)$

$$\frac{1}{1+x} \le n \qquad \forall n \in \mathbb{N}^+ \text{ disp.}$$



$$\iff \quad \frac{1}{1+x} \le 1 \qquad (x>-1)$$

$$\iff \quad 1+x \geq 1$$

$$x \geq 0$$ Ops

Ossia $x > -1$ non può essere maggiorante per A_d

$\Rightarrow$ non ci sono maggioranti $x<0$

$$\Rightarrow \sup A_d = 0 \in A_d \quad (n=1)$$

$$0 = \max A_d$$

Cerco minoranti per A_d $y<0$

$$y \leq -\frac{n-1}{n}$$ $\forall n \in \mathbb{N}^+$ disp

$$y \leq -\left[1-\frac{1}{n}\right]$$

$$y \leq -1+\frac{1}{n}$$

$$\boxed{y+1 \leq \frac{1}{n}}$$ (**) $\forall n \in \mathbb{N}^+$ disp

Noto se $y+1 \leq 0$ ottengo che $(**)$ è banalmente verificata (1° membro negativo mentre secondo membro positivo!)

$\Rightarrow$ $y < -1$ sono tutti minoranti

Se invece $y > -1$ da $(**)$ $(y+1>0)$

Passo ai reciproci

$$\frac{1}{y+1} \geq n \quad \forall n \in \mathbb{N}^+ \text{ dispari}$$

ossia $\mathbb{N}^+$ dispari dovrebbe essere lim. sup $\nexists$

Concludo $(A_d)_* = \{ y \in \mathbb{R} : y \leq -1 \}$

$$\Rightarrow \inf A_d = \max (A_d)_* = -1$$

Conclusione

$$\sup A = \max \{1, 0\} = 1$$

$$(\neq \max A)$$

$$\inf A = \min \left\{ \frac{1}{2}, -1 \right\} = -1$$

Ma $-1 = -\frac{n-1}{n}$ n disp. ?

$$1 = \frac{n-1}{n}$$

$$\cancel{n} = \cancel{n} - 1$$

$$0 = -1$$

NO

$\Rightarrow$ $-1 = \inf A$ ma non $\min A$ ∎

Esercizio Studiare la funzione

$$f(x) = \frac{x}{x^2+1}$$

CE $x^2+1 \neq 0$ vera $\forall x \in \mathbb{R}$

$$\Rightarrow \quad CE = \mathbb{R} = \left]-\infty, +\infty\right[$$

$$\lim_{x \to \pm\infty} \frac{x}{x^2+1} = 0$$

$\Rightarrow \boxed{y=0}$ è Asintoto orizz.

Monotonia di $f(x)$ studio $\boxed{f'(x) \geq 0}$

$$f(x) = \frac{x}{x^2+1}$$

$$f'(x) = \frac{1 \cdot (x^2+1) - x \cdot 2x}{(x^2+1)^2} =$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2} =$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

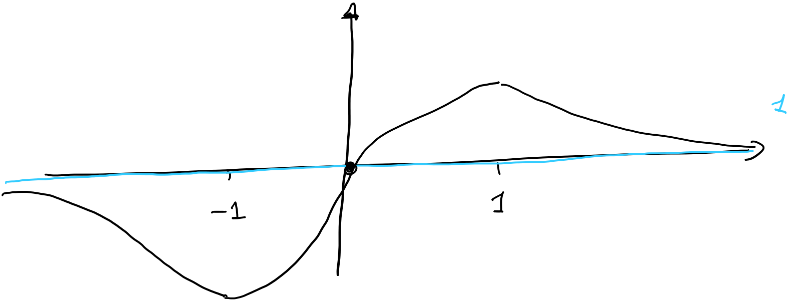
studio $\frac{1-x^2}{(x^2+1)^2} \geq 0$

$\iff 1-x^2 \geq 0$

$\iff x^2-1 \leq 0$

$$\iff \boxed{-1 \le x \le 1}$$

Idea di Grafico



studio $f''(x) \ge 0$

Ricordo $f'(x) = \dfrac{1-x^2}{(1+x^2)^2}$

$$F''(x) = \frac{-2x\cdot(1+x^2)^2 - (1-x^2)\cdot 2(1+x^2)2x}{(1+x^2)^4}$$

$$= \frac{(1+x^2)\cdot 2x}{(1+x^2)^4}\cdot\left[-(1+x^2)-(1-x^2)\cdot 2\right]$$

$$= \underline{(1+x^2)2x}\;\left[-1-x^2-2+2x^2\right]$$

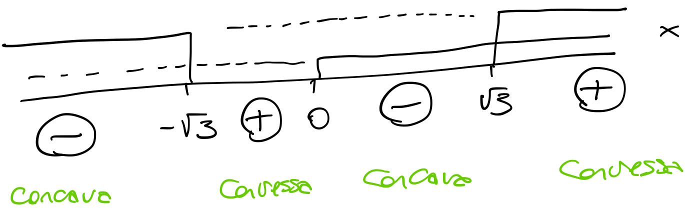
$$(1+x^2)^4$$

$$= \frac{(1+x^2)\,2x}{(1+x^2)^4}\left[x^2-3\right]$$

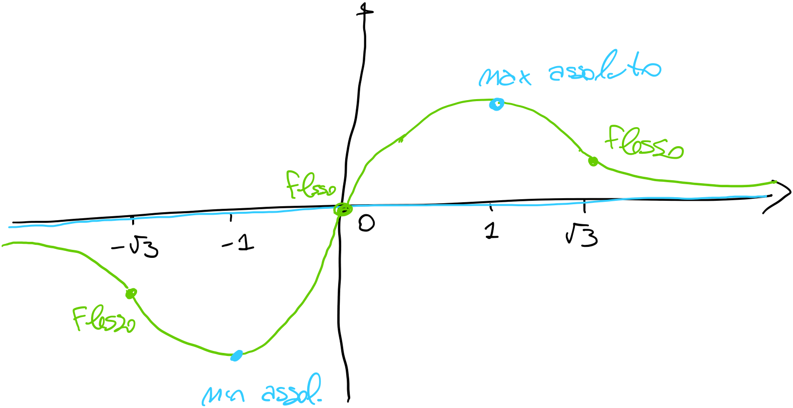
Studio $\boxed{f''(x) \geq 0}$ ossia

$$2\underbrace{\frac{(1+x^2)}{(1+x^2)^4}}_{\substack{>0 \\ \text{sempre}}} \cdot x \cdot (x^2-3) \geq 0$$

$$\iff x(x^2-3) \geq 0 .$$



Disegno un grafico completo



si ha $x = 1$ max assoluto

$x = -1$ min assoluto

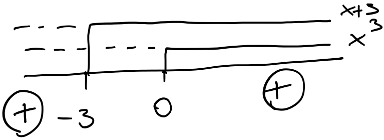
$x = \pm\sqrt{3}, 0$ flessi

Esercizio Studiare la funzione

$$f(x) = \sqrt{\frac{x^3}{x+3}}$$

$CE = \underline{x^3} \geq 0 \quad , \quad x+3 \neq 0$

$x+3$



$$\downarrow\boxed{x \neq -3}$$

$$CE = \left]-\infty, -3\right[\cup \left[0, +\infty\right[$$

$$\lim_{x \to -3^-} \sqrt{\frac{x^3}{x+3}} = \left(\sqrt{\frac{-27}{0^-}}\right) = +\infty$$

$x \to -3^-$ significa $x < -3$ ossia $x+3<0$

$$\lim_{x \to +\infty} \sqrt{\frac{x^3}{x+3}} = +\infty$$

$$\lim_{x \to -\infty} \sqrt{\frac{x^3}{x+3}} = +\infty$$

Studio Asintoti Obliqui

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{1}{x} \cdot \sqrt{\frac{x^3}{x+3}}$$

$$= \lim_{x \to +\infty} \frac{1}{\sqrt{x^2}} \cdot \sqrt{\frac{x^3}{x+3}} =$$

$(x>0)$

$$= \lim_{x \to +\infty} \sqrt{\frac{x^3}{x^2(x+3)}} = 1$$

$$\lim_{x \to +\infty} [F(x) - 1 \cdot x] =$$

$$= \lim_{x \to +\infty} \sqrt{\frac{x^3}{x+3}} - x = \quad (+\infty - \infty)$$

$$= \lim_{x \to +\infty} \left(\sqrt{\frac{x^3}{x+3}} - x\right) \frac{\sqrt{\frac{x^3}{x+3}} + x}{\sqrt{\frac{x^3}{x+3}} + x} =$$

$$= \lim_{x \to +\infty} \frac{\frac{x^3}{x+3} - x^2}{\sqrt{\frac{x^3}{x+3}} + x} =$$

$$= \lim_{x \to +\infty} \frac{\frac{\cancel{x^3} - \cancel{x^3} - 3x^2}{x+3}}{\sqrt{\frac{x^3}{x+3}} + x} =$$

$$= \lim_{x \to +\infty} - \frac{3x^2}{x+3} \cdot \frac{1}{\sqrt{\frac{x^3}{x+3}} + x} =$$

$$= \lim_{x \to +\infty} - \underbrace{\frac{3x^2}{x+3} \cdot \frac{1}{x}}_{3} \cdot \underbrace{\frac{1}{\sqrt{\frac{x}{x+3}} + 1}}_{\frac{1}{2}}$$

$$= -\frac{3}{2}$$

$\Rightarrow \quad y = x - \frac{3}{2}$ è Asintoto Obliquo destro

$$\lim \frac{f(x)}{x} = \lim_{x \to -\infty} \frac{1}{x} \cdot \sqrt{\frac{x^3}{x+3}}$$

$x \to -\infty$

$$= \lim_{x \to -\infty} \frac{1}{-\sqrt{x^2}} \cdot \sqrt{\frac{x^3}{x+3}} =$$

Infatti $\sqrt{x^2} = |x| = -x$ (se $x<0$)

$$= \lim_{x \to -\infty} -\sqrt{\frac{x^3}{x^2(x+3)}} = -1$$

$$\lim_{x \to -\infty} \left[f(x) - (-1) \cdot x \right] =$$

$$= \lim_{x \to -\infty} \sqrt{\frac{x^3}{x+3}} + x \qquad (\infty - \infty)$$

$$= \lim_{x \to -\infty} \left(\sqrt{\frac{x^3}{x+3}} + x \right) \frac{\sqrt{\frac{x^3}{x+3}} - x}{\sqrt{\frac{x^3}{x+3}} - x}$$

$$= \lim_{x \to -\infty} \frac{\frac{x^3}{x+3} - x^2}{\sqrt{\frac{x^3}{x+3}} - x} =$$

$$= \lim_{x \to -\infty} \frac{\dfrac{\cancel{x^3} - \cancel{x^3} - 3x^2}{x+3}}{\sqrt{\dfrac{x^3}{x+3}} - x}$$

$$= \lim_{x \to -\infty} \frac{-3x^2}{x+3} \cdot \frac{1}{\sqrt{\dfrac{x^3}{x+3}} - x} =$$

$$\boxed{x<0} \quad = \lim_{x \to -\infty} -\frac{3x^2}{x+3} \cdot \frac{1}{\sqrt{x^2}\sqrt{\dfrac{x}{x+3}} - x}$$

$$= \lim_{x \to -\infty} -\frac{3x^2}{x+3} \quad \frac{1}{-x\sqrt{\dfrac{x}{x+3}} - x}$$

$$= \lim_{x \to -\infty} \underbrace{\frac{\cancel{-}3x^2}{(x+3)(\cancel{-}x)}}_{3} \cdot \underbrace{\frac{1}{\sqrt{\dfrac{x}{x+3}} + 1}}_{\frac{1}{2}}$$

$$= \frac{3}{2}$$

$\Rightarrow \quad y = -x + \frac{3}{2}$ è Asintoto Obl. sinistro

Monotonia : $F'(x) \geq 0$

$$F(x) = \sqrt{\frac{x^3}{x+3}} = \left(\frac{x^3}{x+3}\right)^{\frac{1}{2}}$$

$$F(x) = h(g(x)) \quad \text{con} \quad h(z) = z^{\frac{1}{2}}$$

$$g(x) = \frac{x^3}{x+3}$$

$$h'(z) = \frac{1}{2} \cdot z^{\frac{1}{2}-1} = \frac{1}{2\sqrt{z}}$$

$$g'(x) = \frac{3x^2(x+3) - x^3}{(x+3)^2} = \frac{2x^3 + 9x^2}{(x+3)^2}$$

$$\Rightarrow \quad F'(x) = \frac{1}{2\sqrt{\frac{x^3}{x+1}}} \cdot \frac{2x^3+9x^2}{(x+3)^2}$$

studio $f'(x) \geq 0$

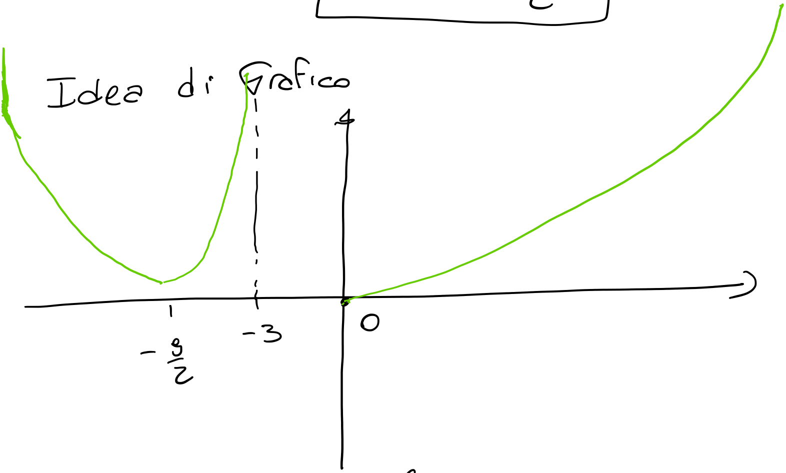
$$\iff \quad 2x^3 + 9x^2 \geq 0$$

$$x^2\,(2x+9) \geq 0$$

$$\iff \quad 2x+9 \geq 0$$

$$\iff \quad \boxed{x \geq -\frac{9}{2}}$$

Idea di Grafico



$x = -\frac{9}{2}$ è min rel

$x = 0$ è min Assoluto

$\boxed{\sup f = +\infty}$

Non ci sono max rel

Esercizio Studiare la Funzione

$$F(x) = \frac{|x|}{1+x^3}$$

CE $1+x^3 \neq 0 \iff \boxed{x \neq -1}$

CE $]-\infty, -1[\cup]-1, +\infty[$

Esplicito il valore assoluto

$$F(x) = \begin{cases} \dfrac{x}{1+x^3} & \text{se } x \geq 0 \\[2ex] -\dfrac{x}{1+x^3} & \text{se } x < 0 \end{cases}$$

$$\lim_{x\to -1^-} F(x) = \lim_{x\to -1^-} -\frac{x}{1+x^3} \quad \left(\frac{1}{0^-}\right)$$

$$= -\infty$$

se $x < -1$

ossia $x^3 < -1$

$\Leftrightarrow \boxed{x^3+1<0}$

$$\lim_{x \to -1^+} \frac{-x}{1+x^3} = \left(\frac{1}{0^+}\right) = +\infty$$

$x = -1$ Asintoto verticale

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x}{1+x^3} = 0$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} -\frac{x}{1+x^3} = 0$$

$\Rightarrow$ $\boxed{y=0 \text{ Asintoto oriz.}}$

Monotonia

$$F'(x) = \begin{cases} \dfrac{1\cdot(1+x^3) - x\,3x^2}{(1+x^3)^2} & \boxed{x>0} \\[2ex] -\left[\dfrac{1+x^3 - 3x^3}{(1+x^3)^2}\right] & \boxed{x<0} \end{cases}$$

$$F'(x) = \begin{cases} \dfrac{-2x^3+1}{(1+x^3)^2} & \boxed{x>0} \\[2ex] \dfrac{2x^3-1}{(1+x^3)^2} & \boxed{x<0} \end{cases}$$

$$\lim_{x\to 0^-} F'(x) = \lim_{x\to 0^-} \frac{2x^3-1}{(1+x^3)^2} = \frac{-1}{1}$$

$$= -1$$

$$\lim_{x\to 0^+} F'(x) = \lim_{x\to 0^+} \frac{1-2x^3}{(1+x^3)^2} = 1$$

$\Rightarrow \boxed{x=0}$ è punto Angoloso

studio $\boxed{F'(x) \geq 0}$

caso $\boxed{x>0}$ $\dfrac{1-2x^3}{(1+x^3)^2} \geq 0$

$$\iff \quad 1-2x^3 \geq 0$$

$$\iff \quad 1 \geq 2x^3$$

$$\iff \quad \frac{1}{2} \geq x^3$$

$$\iff \quad \boxed{\frac{1}{\sqrt[3]{2}} \geq x}$$

caso $\boxed{x<0}$ $$\frac{2x^3-1}{(1+x^3)^2} \geq 0$$

$$\iff \quad 2x^3-1 \geq 0$$

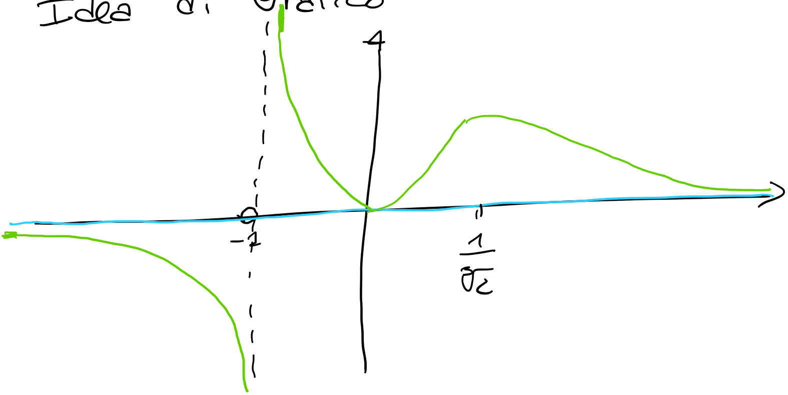
$$\iff \quad x^3 \geq \frac{1}{2}$$

$$\iff \quad x \geq \frac{1}{\sqrt[3]{2}}$$

$\boxed{\text{Mai'}}$

Fcn decresa in $]-\infty, 0[\setminus \{-1\}$

Idea di Grafico



$$F'(x) = \begin{cases} \dfrac{1-2x^3}{(1+x^3)^2} & \text{se } x>0 \\[2ex] \dfrac{2x^3-1}{(1+x^3)^2} & \text{se } x<0 \end{cases}$$

Faccio $F''(x)$:

caso $\boxed{x>0}$

$$F''(x) = \frac{-2\cdot 3x^2(1+x^3)^2 -}{(1+x^3)^4}$$

$$\underline{-(1-2x^3)\,2\,(1+x^3)\cdot 3x^2}$$

$$= \frac{-2\cdot 3\, x^2 (1+x^3)\left[(1+x^3)+(1-2x^3)\right]}{(1+x^3)^4}$$

$$= \underbrace{\boxed{\frac{-6x^2(1+x^3)}{(1+x^3)^4}}}_{<0} \cdot (2-x^3) \geq 0$$

ricordo $\boxed{x>0}$ $\nearrow$

$$\iff 2-x^3 \leq 0$$

$$\iff 2 \leq x^3$$

$$\iff \boxed{\sqrt[3]{2} \leq x}$$

F(x) convessa $[\sqrt[3]{2}, +\infty[$

F(x) concava in $[0, \sqrt[3]{2}[$

caso $\boxed{x<0}$

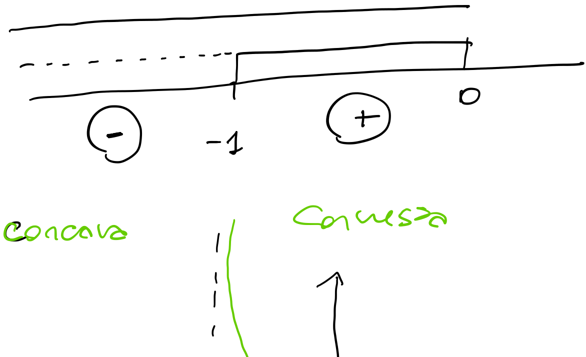
$$F''(x) = \underbrace{\frac{6x^2(1+x^3)}{(1+x^3)^4}}_{\geq 0}(2-x^3) \geq 0$$

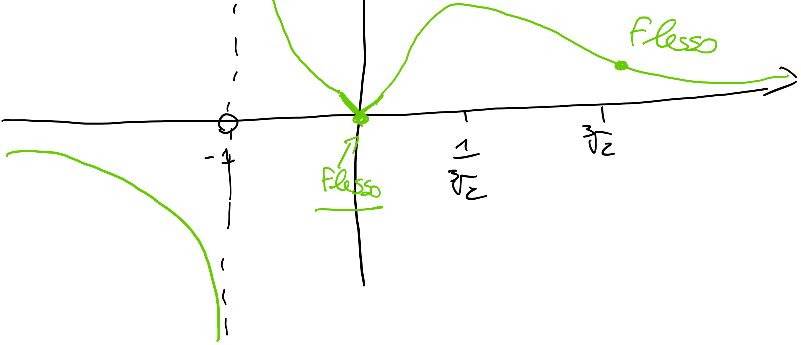
$$\iff (1+x^3)\cdot(2-x^3) \geq 0$$

$$1+x^3 \geq 0 \iff \boxed{x \geq -1}$$

$$2-x^3 \geq 0 \iff x \leq \sqrt[3]{2}$$ sempre positivo

$\boxed{x<0}$





$x = 0, \sqrt[3]{2}$ Flessi

$x = \frac{1}{\sqrt[3]{2}}$ max rel

$x = 0$ min rel

$\sup f = +\infty$ $\inf f = -\infty$